

Physics 128 Discussion (Week 7)

Derivation of Friedmann equations

Einstein's field equations with a cosmological constant Λ are given by:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (1)$$

where:

- $G_{\mu\nu}$ is the Einstein tensor describing spacetime curvature.
- $T_{\mu\nu}$ is the energy-momentum tensor of matter.
- Λ is the cosmological constant.

For a homogeneous and isotropic universe, we use the FLRW metric in the $(+, -, -, -)$ convention

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (2)$$

where $a(t)$ is the scale factor and k is the curvature parameter. The energy-momentum tensor for a perfect fluid is:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu} \quad (3)$$

where ρ is the energy density and P is the pressure. For a comoving observer

$$T_{00} = \rho, \quad T_{ii} = -P g_{ii}. \quad \left(\because U_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \quad (4)$$

The Einstein tensor is given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \quad (5)$$

For the FLRW metric, the relevant components are:

$$G_{00} = 3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right), \quad (6)$$

$$G_{ii} = - \left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) g_{ii}. \quad (7)$$

Thus we get

$$\begin{aligned} 3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) &= 8\pi G \rho + \Lambda \\ \Rightarrow \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} &= \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}. \end{aligned} \quad (8)$$

Defining the Hubble parameter $H = \dot{a}/a$, we obtain:

$$H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} - \frac{k}{a^2}. \quad (9)$$

Using $G_{ii} + \Lambda g_{ii} = 8\pi G T_{ii}$:

$$- \left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = -8\pi G p + \Lambda. \quad (10)$$

Rearrange:

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -8\pi G p. \quad (11)$$

Using the first Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (12)$$

From $\nabla^\mu T_{\mu\nu} = 0$, we obtain the continuity equation:

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (13)$$

The fundamental equations governing an FLRW universe with a cosmological constant are:

- **First Friedmann Equation (Expansion Rate):**

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}. \quad (14)$$

- **Second Friedmann Equation (Acceleration):**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (15)$$

- **Continuity Equation (Energy Conservation):**

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (16)$$

These equations describe the evolution of the universe's expansion including the effect of dark energy.

Problem 1 (Einstein's Static Universe)

Einstein, in an attempt to construct a static model for the Universe, introduced the cosmological constant $\Lambda > 0$.
(a) What is the relation between Λ and the matter density ρ for a static Universe in which the energy density of relativistic matter is negligible?

Let us start from the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (17)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (18)$$

In Einstein's static universe, the scale factor is constant, so both

$$\dot{a} = 0, \quad \ddot{a} = 0. \quad (19)$$

When the relativistic matter is negligible, we only need to think of the non-relativistic matter (dust) besides Λ . For non-relativistic matter, the pressure is negligible ($p \simeq 0$), and since $\ddot{a} = 0$ we obtain

$$0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}. \quad (20)$$

Thus, the cosmological constant Λ must satisfy

$$\Lambda = 4\pi G\rho. \quad (21)$$

(b) Show that Einstein's static universe has to be positively curved (i.e., $k = +1$), and demonstrate that the present-day value for the scale-factor, a_0 (also called the radius of curvature) has to be equal to $\Lambda^{-1/2}$.

From the acceleration equation, we have

$$\rho = \frac{\Lambda}{4\pi G}. \quad (22)$$

Plug this result into the static Friedmann equation for $\dot{a} = 0$

$$\begin{aligned} 0 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\ &= \frac{8\pi G}{3}\left(\frac{\Lambda}{4\pi G}\right) - \frac{k}{a^2} + \frac{\Lambda}{3} \\ &= \Lambda - \frac{k}{a^2}. \end{aligned} \quad (23)$$

Thus we get

$$k = \Lambda a^2 > 0. \quad (24)$$

In the standard normalization for the Friedmann equation, k is taken to be one of $-1, 0$, or $+1$. Then we must have

$$k = +1, \quad (25)$$

which means the universe is positively curved (*i.e.* closed). Moreover, if we denote the present-day (or static) scale factor by a_0 , the equation becomes

$$\Lambda = \frac{1}{a_0^2}, \quad (26)$$

or equivalently

$$a_0 = \Lambda^{-1/2}. \quad (27)$$

(c) Consider once more Einstein's static universe. Suppose that some matter is converted to radiation (for example, due to stars and/or active galactic nuclei). What happens to the scale factor of the universe? Explain your answer.

Now, if some matter is converted into radiation, although the total energy density might remain (approximately) the same, the effective gravitational “charge” changes. Recall that for a perfect fluid the source of gravity in the acceleration equation is not just ρ but rather $\rho + 3p$. For non-relativistic matter (dust) $p_{\text{matter}} \simeq 0$, so the contribution is just ρ_{matter} . For radiation, however, $p_{\text{rad}} = \rho_{\text{rad}}/3$, so the contribution becomes

$$\begin{aligned}\rho_{\text{rad}} + 3p_{\text{rad}} &= \rho_{\text{rad}} + 3\left(\frac{\rho_{\text{rad}}}{3}\right) \\ &= 2\rho_{\text{rad}}.\end{aligned}\tag{28}$$

Thus, converting some matter into radiation increases the effective gravitational attraction (per unit energy density). The balance that was achieved in the static universe is upset because the cosmological constant Λ was originally tuned to cancel the gravitational pull of dust. With radiation present, the extra pressure causes the attractive term (now $\propto \rho_{\text{matter}} + 2\rho_{\text{rad}}$) to be larger than what Λ can counteract. As a result, the net effect is that the universe is no longer in equilibrium. The additional gravitational pull makes the acceleration equation negative

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_{\text{matter}} + 2\rho_{\text{rad}}) + \frac{\Lambda}{3} < 0.\tag{29}$$

This negative acceleration causes the scale factor to begin decreasing, in other words, the universe will start to contract. In summary, converting matter to radiation in Einstein’s static universe destroys the delicate balance between the gravitational attraction and the repulsive effect of Λ , leading to a contraction of the universe (a decrease in the scale factor).

(d) Suppose the energy density associated with the cosmological constant is equal to the present-day critical density. What is the total energy associated with Λ within 1 AU, in units of the rest-mass energy of the Sun? What does this imply for the impact of Λ on the the dynamics of the Solar system?

We begin by noting that if the energy density of the cosmological constant (or dark energy) is equal to the present-day critical density, then

$$\begin{aligned}\rho_{\Lambda} &= \rho_{\text{crit}}^0 \\ &= \frac{3H_0^2}{8\pi G} \\ &\sim 10^{-29} \text{ g cm}^{-3}.\end{aligned}\tag{30}$$

The total mass contained within a sphere of radius $R = 1 \text{ AU}$ ($\approx 1.5 \times 10^{13} \text{ cm}$) is given by

$$\begin{aligned}M_{\Lambda} &= \rho_{\Lambda} V_{\text{AU}} \\ &= \rho_{\Lambda} \left(\frac{4\pi}{3} R^3\right) \\ &\sim (10^{-29} \text{ g cm}^{-3}) (1.41 \times 10^{40} \text{ cm}^3) \\ &= 1.41 \times 10^{11} \text{ g}.\end{aligned}\tag{31}$$

Thus

$$\begin{aligned}\frac{M_{\Lambda}}{M_{\odot}} &\sim \frac{1.41 \times 10^{11} \text{ g}}{2 \times 10^{33} \text{ g}} \\ &= 7 \times 10^{-23}.\end{aligned}\tag{32}$$

This exceedingly small ratio shows that the energy density of the cosmological constant has an utterly negligible effect on the dynamics within the solar system. In other words, the gravitational influence (or “anti-gravity” repulsion) of dark energy is many orders of magnitude too weak on these small scales to affect the orbits of the planets or the overall gravitational binding of the solar system. Therefore, despite the cosmological constant being of crucial importance on cosmological scales, its impact on solar system dynamics is completely insignificant even in the Λ -dominated universe.